II. Mass and Force in Elementary Dynamics.

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In a series of articles published during the last few years, Professor Huntington has contended for the use of the equation

$$F/F' = a/a',$$

instead of the equation

$$ma = \lambda F,$$

as the fundamental equation of dynamics. A reader of his paper entitled "The logical skeleton of elementary dynamics," recently published in the Monthly,[1] can scarcely question the correctness and adequacy of his treatment. Controversy can have to do only with matters of arrangement and emphasis, on which anyone may naturally have an opinion. It has been my peculiar privilege to discuss the subject informally with Professor Huntington at frequent intervals, and I may perhaps be allowed to acknowledge my indebtedness to him for a very great clarification of my own ideas, and at the same time to say a few words with regard to some of the points on which I have remained unconvinced. To novelty in the views presented I can make no claim; I offer them here merely for purposes of ready comparison.

It is characteristic of Professor Huntington's presentation that the notion of *mass* is subordinated to the notion of *force*, and is not merely assigned a secondary place at the beginning, but is regarded as of secondary importance throughout. It manifests itself as *inertia*, through the constant ratio F/a of the fundamental equation, and can be measured in terms of *standard weight*. All that is needed for the solution of problems is contained in these two ideas. The reference of all measurements to fundamental units of force, length, and time, does away with the difficulties that arise when units of mass and units of force are used together in such a way that it is necessary to consider the relations between them. The whole approach to dynamical theory is extraordinarily simplified.

It may seem that with this acknowledgement, discussion should stop. And yet, many teachers and students of mechanics feel that the whole story has not been told, that there was something more in the old idea of mass, which was altogether worth while when you understood it. Granted that it is best to begin with the equation in Professor Huntington's form, and not to pile up difficulties at the threshold of the subject, there ought to come a time when you are ready to search the concept of mass for all that there is in it.[2] The skeleton of elementary dynamics may have every bone in place, but, like other

[1] American Mathematical Monthly, Vol. 24 (1917), pp. 1–16. See also references to controversial articles in *Science*, at the close of the paper.

[2] The writer, who has had occasion to teach an elementary course in mechanics for several years, has tried two or three times the experiment of using the equation $F/F' = a/a'$ for perhaps the first three weeks of kinetics, and then introducing the equation $ma = \lambda F$, with a discussion of units of force and mass; and he is at present disposed to think that any lost motion that results is more than compensated by the advantages of the arrangement.

skeletons, it loses in richness of outline something of what it gains in transparency. So far as I can judge my own sentiments by introspection, the pleasure that I take in the use of the equation $F = ma$, with all its pounds and poundals, has nothing of pedantry about it; it is as genuine and healthy as that which belongs to the use of any improved tool in analysis.

The study of mechanics in college serves two purposes. It leads to the solution of practical or conceivably practical problems, and it stimulates the imagination by an insight into the orderly working of the universe. Few students will ever have any "use" for any but the terrestrial applications of the general laws; but neither the historical development of the science nor its far-reaching significance can be understood without reference to the working of its laws among the heavenly bodies. While the equation $F/F' = a/a'$ can with entire justification be taken as the fundamental one, there as elsewhere, still there is reason for regarding something else as equally fundamental at the same time.

Two things are characteristic of material bodies throughout the universe. They are disposed to disturb other bodies, and they are reluctant to be disturbed by them. It is highly remarkable that each body, wherever it goes, and to whatever physical or chemical changes it is subjected, has a characteristic quantity invariably associated with it, which measures its power of attracting other bodies, in accordance with the law of universal gravitation. It is highly remarkable that each body has a quantity invariably associated with it which measures its inertia, in accordance with the equation $F/F' = a/a'$. Most remarkable of all, however, is the circumstance that for different bodies these two "body-constants" stand in an invariable relation to each other. Every material body has the property of attractiveness and the property of inertia, and these are somehow not two properties, but one. This, it seems to me, is the significance of the word *mass* —it is a name for the one fundamental property which manifests itself in two aspects. The equation $F/F' = a/a'$ describes one aspect; the equation $ma = \lambda f$ describes one, implies the other, and asserts the essential identity of the two. The second equation, admittedly more difficult, is richer in content and more suggestive. Even to supplement the other by the statement that inertia is proportional to standard weight, does not produce quite the same effect, for it is more impressive to invoke the law of universal gravitation at the start than merely to observe that at a particular station on the earth's surface different bodies fall with the same acceleration, and mention the general law only incidentally later. The idea of mass, and the equation containing it, are worthy of all the emphasis that it is possible to give them.

The contention, sometimes advanced, that the idea of inertia is more fundamental than the other part of the mass-concept, that without bringing in acceleration it is impossible to form any precise quantitative notion of mass at all, does not seem to me well sustained. Of course the first rough idea of a beam-balance for measuring mass in terms of its gravitational power has to be refined, but so does the naïve conception of a spring-balance and a fixed frame of reference for measuring force and acceleration. There is no difficulty in either case if you are

not too critical; and if you are, the difficulties in either case are great, if not for the present insurmountable. The two are very much on a par.

So far, nothing has been said about the interpretation of mass as quantity of matter. As Professor Huntington has insisted, this attribute can hardly be made the basis for a satisfactory *definition* of mass, for the purposes of dynamical theory. But as a bond between that theory, founded on suitable definitions, and the facts of every-day experience, it is undoubtedly important to recognize that mass *is* quantity of matter, so far as quantity of matter means anything at all, and this recognition adds very much to the significance of mass. Not only does a quart of water have twice the mass of a pint, but if you take a quart of water and compress it under great pressure, or boil it, or freeze it, or decompose it by electrolysis, it still has the same mass as before. The qualities of mass seem somehow inherent in the matter itself, inseparable from it when it is otherwise changed in almost every conceivable way. It is to be said that the primary appeal of the identification of mass with quantity of matter is perhaps cruder still: you measure mass on a beam-balance, and a beam-balance, in real life, is an instrument for finding out how much of a substance you have got.

Leaving quantity of matter aside, we have made the concept of mass, fundamental as it is, depend logically on the concept of force; for the latter enters both into the statement of the law of gravitation and into the statement of the law of inertia. But it is not to be granted that the former concept is even to this extent essentially a subordinate one. It is perfectly possible to introduce the two side by side. For the measurement of masses by means of a beam-balance does not necessarily involve the measurement of force—quantitative determination of the relative magnitudes of different forces—nor even the concept of force as a measurable quantity. If it seems necessary to make the measurement of mass depend on the existence of force as a physical agency, the difference is that force is recognizably homogeneous, and for that reason capable of independent measurement, while matter is not; quantity of force has just the *a priori* significance which it is hard to attach to quantity of matter.

Finally, as to the choice of fundamental units for a systematic table, it is certainly allowable to start with mass, length, and time, and define force in terms of them, and then to define anything you please in terms of force, length, and time, and write its dimensions accordingly. The dimensions of work are FL, whether you remember the equation $F = MLT^{-2}$ or forget it or never had it to forget. There is a certain loss of formal simplicity in permitting a derived unit to act as deputy for a fundamental one, but it is not such a loss as to put any burden on the understanding; and it is merely a question whether it is worth while to make so slight a sacrifice for the sake of commemorating in the table the fact that the unit of force is derived from the unit of mass in practice.